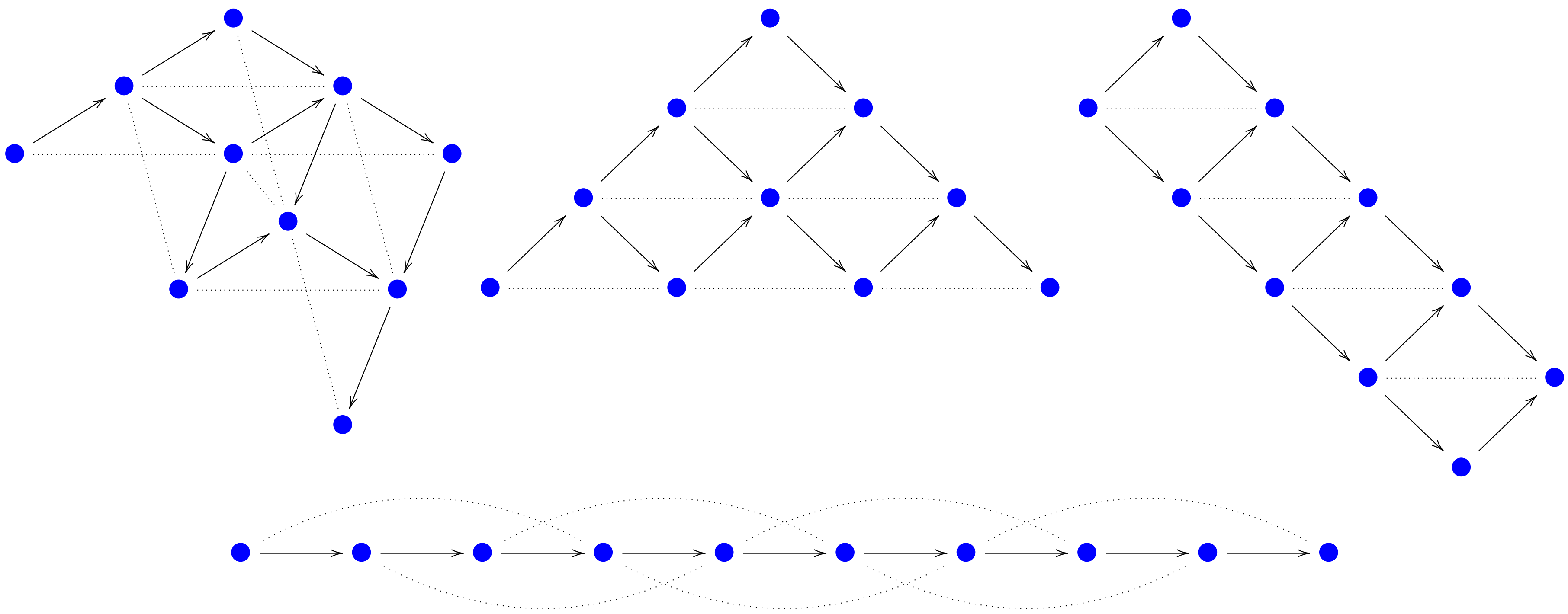
Combinatorial aspects of derived equivalence

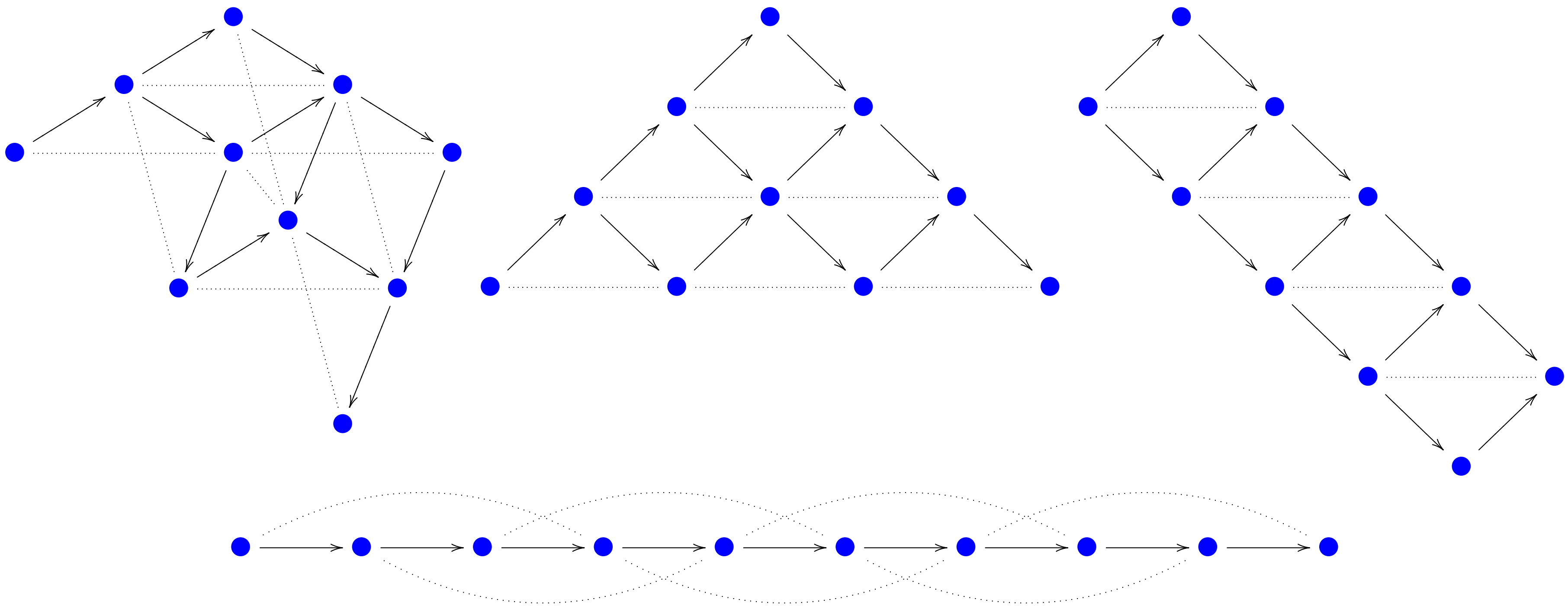
Sefi Ladkani

University of Bonn

http://guests.mpim-bonn.mpg.de/sefil/

What is the connection between . . .





The *finite dimensional algebras* arising from these combinatorial data given by *quivers with relations* have equivalent *derived categories* of modules.

Quivers with relations

A *quiver* Q is an oriented graph.

K – field, the *path algebra* KQ is

- spanned by all paths in Q,
- with multiplication given by composition of paths.

Example.

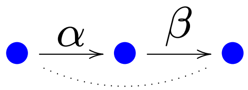
$$Q = \bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3 \qquad KQ = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$$e_1, e_2, e_3, \alpha, \beta, \alpha\beta \qquad \alpha \cdot \beta = \alpha\beta \quad \beta \cdot \alpha = 0$$

Quivers with relations (continued)

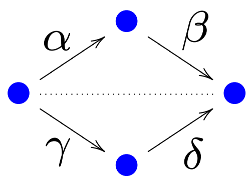
relation – a linear combination of paths having the same endpoints.

- *zero* relation p



$\alpha\beta$

- *commutativity* relation $p - q$



$\alpha\beta - \gamma\delta$

A *quiver* Q *with relations* defines an algebra KQ/I by considering the path algebra KQ modulo the ideal I generated by all the relations.

Theorem [Gabriel]. If K is algebraically closed, then any finite dimensional K-algebra is Morita equivalent to a quiver with relations.

Example 1 – Line

K – field, $n, r \geq 2$,

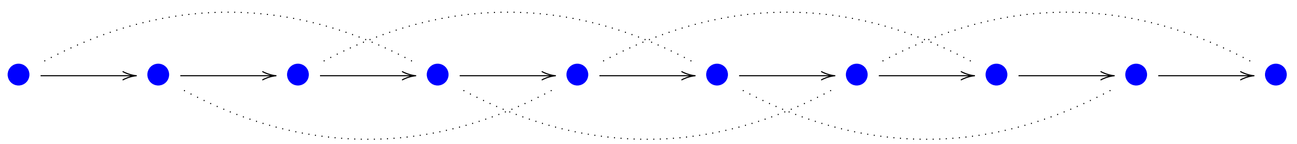
$$Line(n,r) = K\overrightarrow{A_n}/(x^r)$$

Given by the *linear* quiver $\overrightarrow{A_n}$

$$\bullet_1 \xrightarrow{x} \bullet_2 \xrightarrow{x} \bullet_3 \xrightarrow{x} \cdots \xrightarrow{x} \bullet_n$$

with *zero relations* – all the paths of length r.

Example. $Line(10,3)$

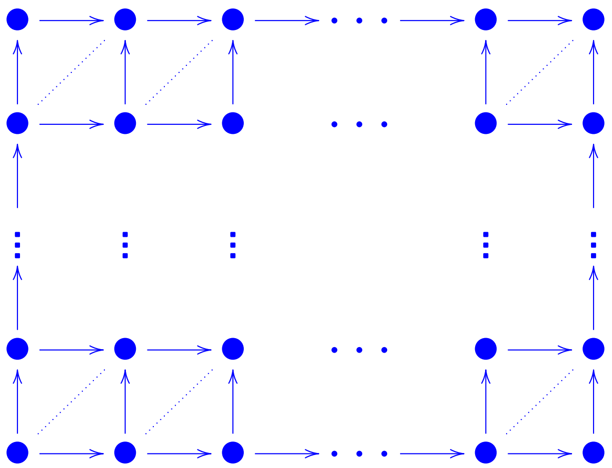


Example 2 – Rectangle

$n, m \geq 1$.

$$Rect(n,m) = K\overrightarrow{A_n} \otimes_K K\overrightarrow{A_m}$$

Given by the *rectangular* n-by-m quiver



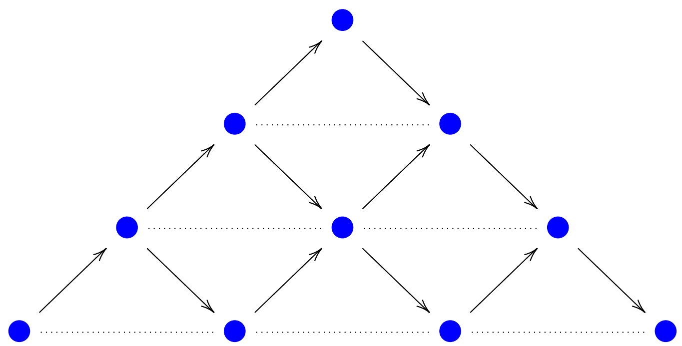
with all *commutativity relations*.

Example 3 – Triangle

$Triang(n)$ is the *Auslander algebra* of $K\overrightarrow{A_n}$.

It has a *triangular* quiver having sides of length n, with *zero* and *commutativity* relations.

Example. $Triang(4)$



Derived categories

$\mathcal{A}$ – abelian category, $\mathcal{C}(\mathcal{A})$ – the category of *complexes*

$$K^\bullet = \dots \xrightarrow{d} K^{-1} \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} \dots$$

with $K^i \in \mathcal{A}$ and $d^2 = 0$.

A morphism $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if

$$H^i f : H^i K^\bullet \to H^i L^\bullet$$

are isomorphisms for all $i \in \mathbb{Z}$.

The *derived category* $\mathcal{D}(\mathcal{A})$ is obtained from $\mathcal{C}(\mathcal{A})$ by *localization* with respect to the quasi-isomorphisms (that is, we formally invert all quasi-isomorphisms). It is a *triangulated category*.

Perspective

Triangulated and *derived categories* can relate objects of different nature:

- *Coherent sheaves* over algebraic varieties and *modules* over non-commutative algebras [Beilinson 1978, Kapranov 1988]
- *Homological mirror symmetry conjecture* [Kontsevich 1994]

. . . but also relate non-isomorphic objects of the same nature:

- Morita theory for derived categories of modules [Rickard 1989]
- Derived categories of coherent sheaves [Bondal-Orlov 2002]
- *Broué's conjecture* on blocks of group algebras [Broué 1990]

Derived equivalence of rings

Theorem [Rickard 1989]. Let R, S be rings. Then

$$\mathcal{D}(\mathrm{Mod}\, R) \simeq \mathcal{D}(\mathrm{Mod}\, S) \qquad (R, S \text{ are } \textit{derived equivalent}, \ R \sim S)$$

if and only if there exists a *tilting complex* $T \in \mathcal{D}(\mathrm{Mod}\, R)$

- *exceptional*: $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\, R)}(T, T[i]) = 0$ for $i \neq 0$,
- *compact generator*: $\langle \mathrm{add}\, T \rangle = \mathrm{per}\, R$,

such that $S \simeq \mathrm{End}_{\mathcal{D}(\mathrm{Mod}\, R)}(T)$.

Problems. existence? constructions?

Derived equivalences of lines, rectangles and triangles

Theorem [L].

$$Rect(n, r) \sim Line(n \cdot r, r + 1)$$
$$Rect(2r + 1, r) \sim Triang(2r)$$

Example. $Line(10, 3) \sim Rect(5, 2) \sim Triang(4)$.

Remark. Can be generalized to higher dimensional shapes (simplices, prisms, boxes etc.)

- Derived *accessible algebras* [Lenzing - de la Peña 2008]
- Categories of singularities; *weighted projective lines*; nilpotent operators [Kussin-Lenzing-Meltzer]
- Higher *ADE chain*.

Tilting complexes from existing ones – tensor

A, B – K-algebras, K – commutative ring, $\otimes = \otimes_K$,
T – tilting complex over A,
U – tilting complex over B + *technical conditions* . . .

Theorem [Rickard 1991]. $T \otimes U$ is a *tilting complex* over $A \otimes B$ with endomorphism ring $\mathrm{End}_{\mathcal{D}(A)}(T) \otimes \mathrm{End}_{\mathcal{D}(B)}(U)$. Hence

$$A \otimes B \sim \mathrm{End}_{\mathcal{D}(A)}(T) \otimes \mathrm{End}_{\mathcal{D}(B)}(U).$$

Remark. Derived equivalence between *tensor products* of algebras.

New tilting complexes from existing ones

$T_1, T_2, \dots, T_n$ – tilting complexes over A,
$U_1 \oplus U_2 \oplus \dots \oplus U_n$ – tilting complex over B + *technical conditions* ...

Theorem [L]. Assume *multiple exceptionality*:

$$\forall 1 \leq i, j \leq n \quad \text{Hom}_{\mathcal{D}(B)}(U_i, U_j) \neq 0 \Rightarrow \text{Hom}_{\mathcal{D}(A)}(T_i, T_j[r]) = 0 \quad \forall r \neq 0.$$

Then $(T_1 \otimes U_1) \oplus (T_2 \otimes U_2) \oplus \dots \oplus (T_n \otimes U_n)$ is a *tilting complex* over $A \otimes B$ with endomorphism ring given as the *generalized matrix ring*

$$\begin{pmatrix} & \vdots & \\ \dots & M_{ij} & \dots \\ & \vdots & \end{pmatrix}, \text{ where } M_{ij} = \text{Hom}_{\mathcal{D}(A)}(T_j, T_i) \otimes \text{Hom}_{\mathcal{D}(B)}(U_j, U_i).$$

- Derived equivalence between *componentwise tensor products*.
- Implies the derived equivalences of lines, rectangles, triangles ...

Global vs. local operations

The previous derived equivalences are *global* in nature – they change the quiver drastically.

Motivated by an *algorithmic* point of view, we seek *local operations* on the quivers that will produce derived equivalent algebras.

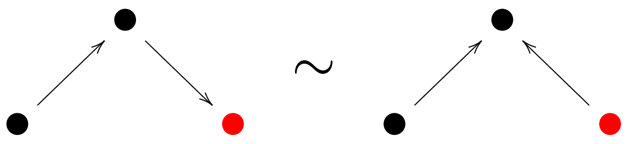
Example – BGP Reflections at sinks/sources

Q – quiver without oriented cycles,
s – *sink* in Q, i.e. no outgoing arrows from s.

$\sigma_s Q$ – the *BGP reflection* with respect to s, obtained from Q by inverting all arrows incident to s, so that s becomes a *source*.

Theorem [Bernstein-Gelfand-Ponomarev]. $KQ \sim K\sigma_s Q$.

Example.



Remark. Generalized by [Auslander-Platzeck-Reiten] to sinks in quivers of arbitrary finite-dimensional algebras.

What about other vertices?

- Combinatorial answer: *quiver mutation* [Fomin-Zelevinsky 2002].
- Algebraic answer: *mutations of algebras*.

We will define these notions and explore the relations between them.

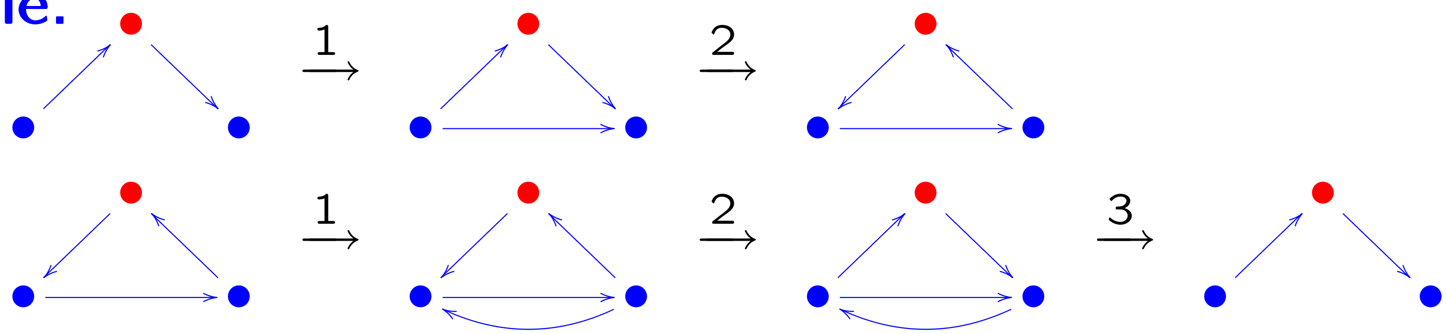
Quiver mutation [Fomin-Zelevinsky]

Q – quiver without *loops* () and *2-cycles* (),
k – any vertex in Q.

The *mutation* of Q at k, denoted $\mu_k(Q)$, is obtained as follows:

1. For any pair $i \xrightarrow{\alpha} k \xrightarrow{\beta} j$, add new arrow $i \xrightarrow{[\alpha\beta]} j$,
2. Invert the incoming and outgoing arrows at k,
3. Remove a maximal set of 2-cycles.

Example.



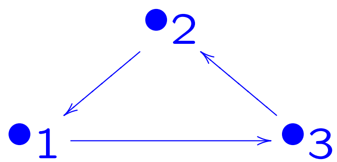
Quivers and anti-symmetric matrices

{*quivers*, no loops and 2-cycles} $\leftrightarrow$ {*anti-symmetric* integral *matrices*}

$$Q \leftrightarrow B_Q$$

$$(B_Q)_{ij} = |\{\text{arrows } j \to i\}| - |\{\text{arrows } i \to j\}|$$

Example.



$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Quiver mutation – matrix version

Mutation as a change-of-basis for the anti-symmetric bilinear form [FZ, Geiss-Leclerc-Schröer]

$$B_{\mu_k(Q)} = (r_k^+)^T B_Q r_k^+ = (r_k^-)^T B_Q r_k^-$$

where

$$r_k^- = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ * & * & -1 & * & * \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad r_k^+ = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ * & * & -1 & * & * \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$(r_k^-)_{kj} = |\{\text{arrows } j \to k\}| \quad (r_k^+)_{kj} = |\{\text{arrows } k \to j\}| \quad (j \neq k)$$

From vertices to complexes

K – algebraically closed field,
$A = KQ/I$ – quiver with relations,

vertex $i \rightsquigarrow$ projective P_i, arrow $i \to j \rightsquigarrow$ map $P_j \to P_i$

k – vertex in Q without loops,

$$T_k^- = \Big(P_k \to \bigoplus_{j \to k} P_j\Big) \oplus \bigoplus_{i \neq k} P_i, \qquad T_k^+ = \Big(\bigoplus_{k \to j} P_j \to P_k\Big) \oplus \bigoplus_{i \neq k} P_i$$

Are these *tilting complexes?*

- Always *compact generators*,
- *Exceptionality* is expressed in terms of the combinatorial data.

Mutations of algebras

If T_k^- is a tilting complex, the *negative mutation* at k is defined as

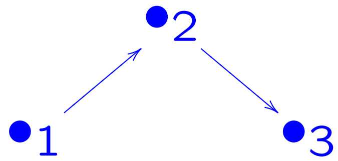
$$\mu_k^-(A) = \mathrm{End}_{\mathcal{D}(A)}(T_k^-)$$

If T_k^+ is a tilting complex, the *positive mutation* at k is defined as

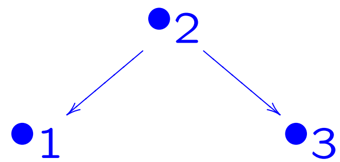
$$\mu_k^+(A) = \mathrm{End}_{\mathcal{D}(A)}(T_k^+)$$

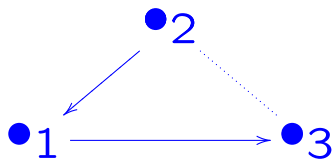
- There are *up to two* mutations at a vertex,
- Mutations yield *derived equivalent* algebras,
- Mutations are *perverse Morita equivalences* [Chuang-Rouquier],
- Closely related to the *Brenner-Butler* tilting modules.

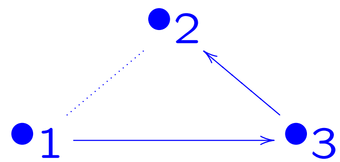
Mutations of algebras – Example

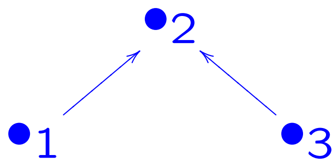
$A =$ 

$\mu_1^-(A)$ is not defined

$\mu_1^+(A) =$ 

$\mu_2^-(A) =$ 

$\mu_2^+(A) =$ 

$\mu_3^-(A) =$ 

$\mu_3^+(A)$ is not defined

Remark. For $A' = \mu_2^-(A)$, neither $\mu_1^-(A')$ nor $\mu_1^+(A')$ are defined.

Cartan matrices and Euler forms

C_A – the *Cartan matrix* of A, defined by $(C_A)_{ij} = \dim_K \mathrm{Hom}_A(P_i, P_j)$.

Remark. The *bilinear form* defined by C_A is invariant under derived equivalence.

Lemma.

$$C_{\mu_k^-(A)} = r_k^- C_A (r_k^-)^T \qquad C_{\mu_k^+(A)} = r_k^+ C_A (r_k^+)^T$$

whenever the mutations are defined.

When A has *finite global dimension*, its *Euler form* is $c_A = C_A^{-T}$, and

$$c_{\mu_k^-(A)} = (r_k^-)^T c_A r_k^- \qquad c_{\mu_k^+(A)} = (r_k^+)^T c_A r_k^+$$

whenever the mutations are defined.

Applications of mutations of algebras

Mutations behave particularly well for the following classes of algebras:

- Algebras of *global dimension 2*
- *2-CY-tilted* algebras, i.e. endomorphism algebras of *cluster-tilting* objects in *2-Calabi-Yau* triangulated categories, including *cluster-tilted* algebras and finite-dimensional *Jacobian* algebras.
  [Amiot, Buan-Iyama-Reiten-Scott, Buan-Marsh-Reineke-Reiten-Todorov, BMR, Iyama-Yoshino, Keller-Reiten, . . .]
- Endomorphism algebras of cluster-tilting objects in *stably 2-CY Frobenius* categories [BIRSc, GLS, Palu, . . .]

Application 1 – Algebras of global dimension 2

A – finite-dimensional K-algebra of *global dimension 2*.

The ordinary quiver Q_A has

$$|\{\text{arrows } i \to j\}| = \dim_K \operatorname{Ext}^1_A(S_i, S_j)$$

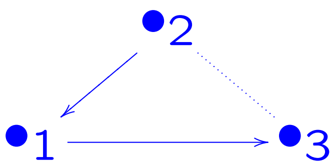
The *extended quiver* $\tilde{Q}_A$ [Assem-Brüstle-Schiffler, Keller] has

$$|\{\text{arrows } i \to j\}| = \dim_K \operatorname{Ext}^1_A(S_i, S_j) + \dim_K \operatorname{Ext}^2_A(S_j, S_i)$$

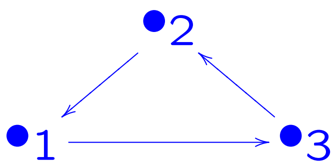
so that $B_{\tilde{Q}_A} = c_A - c_A^T$ is the *anti-symmetrization* of c_A.

Example.

$A =$



$\tilde{Q}_A =$



Mutations of algebras of global dimension 2

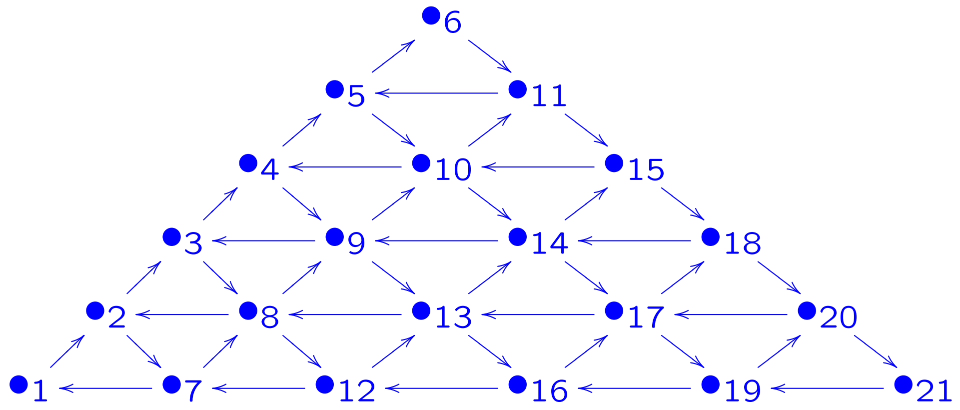
Assume: $\operatorname{gl.dim} A \leq 2$ and $\tilde{Q}_A$ without loops and 2-cycles.

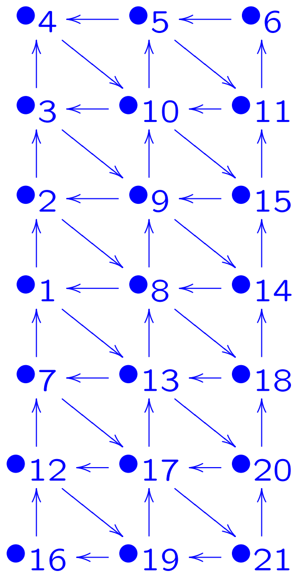
Theorem [L].
If $\mu_k^-(A)$ is defined and $\operatorname{gl.dim} \mu_k^-(A) \leq 2$, then $\tilde{Q}_{\mu_k^-(A)} = \mu_k(\tilde{Q}_A)$.
If $\mu_k^+(A)$ is defined and $\operatorname{gl.dim} \mu_k^+(A) \leq 2$, then $\tilde{Q}_{\mu_k^+(A)} = \mu_k(\tilde{Q}_A)$.

Remark. Not all quiver mutations correspond to algebra mutations.

Question.
Can derived equivalences be realized as sequences of mutations?

Example – Sequence of mutations





1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 1, 2, 3, 4, 5, 12, 13, 14, 15, 7, 8, 9, 10, 1, 2, 3, 4, 21, 19, 16, 20, 17, 12, 18, 13, 7, 21, 19, 16, 20, 17, 12, 21, 19, 16

Consequences for cluster algebras

Theorem [L]. $\widetilde{Q}_{Triang(2r)}$ and $\widetilde{Q}_{Rect(2r+1,r)}$ are *mutation equivalent*.

These are the *cluster types* of the cluster algebra structures on . . .

- $\widetilde{Q}_{Triang(2r)} \rightsquigarrow$ upper-triangular *unipotent matrices* in SL_{2r+2} [Geiss-Leclerc-Schröer]
- $\widetilde{Q}_{Rect(2r+1,r)} \rightsquigarrow$ *Grassmannian* $\mathrm{Gr}_{r+1,3r+3}$ [Scott 2006]

Corollary. These cluster algebras have the same cluster type.

Application 2 – Cluster-tilted algebras

Q – quiver, which is *mutation equivalent* to an *acyclic* one,

Λ_Q – the *cluster-tilted algebra* [BMR] corresponding to Q.

It is the endomorphism algebra of a suitable *cluster-tilting object* in a *cluster category* [BMRRT].

- The quiver of Λ_Q is Q,
- The relations are uniquely determined, using mutations of *quivers with potential* [Derksen-Weyman-Zelevinsky, Buan-Iyama-Reiten-Smith].

Good and bad (quiver) mutations

Motivation. Relate mutation of quivers with mutation of algebras.

The quiver mutation of Q at k is *good* if

$$\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q), \qquad \big(\text{equivalently, } \Lambda_Q \simeq \mu_k^+(\Lambda_{\mu_k(Q)})\big)$$

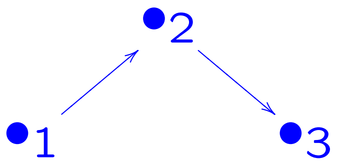
otherwise it is *bad*.

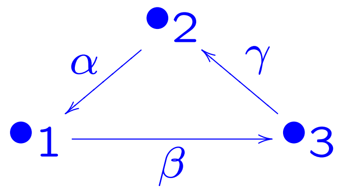
Two reasons for bad quiver mutations:

- The algebra mutation $\mu_k^-(\Lambda_Q)$ is not defined, or
- The algebra mutation $\mu_k^-(\Lambda_Q)$ is defined, but takes incorrect value.

Good and bad mutations – Examples

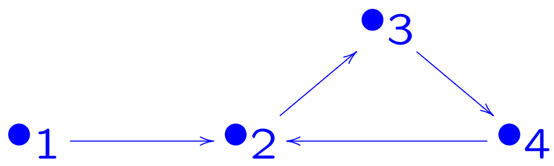
Example. The mutation at the vertex 2 is *bad*.

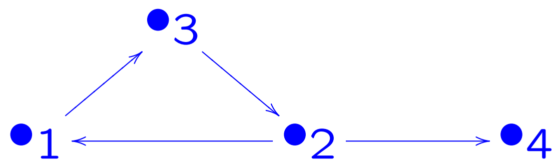




$\alpha\beta, \beta\gamma, \gamma\alpha$

Example. The mutation at the vertex 2 is *good*.





Question. Are "most" mutations good or bad?

Cluster-tilted algebras of Dynkin type E

Theorem [Bastian-Holm-L]. Complete derived equivalence classification of cluster-tilted algebras of Dynkin type E.

The following are equivalent for two such algebras:

- Their Cartan matrices represent *equivalent bilinear forms* over $\mathbb{Z}$,
- They are *derived equivalent*,
- Their quivers can be connected by a *sequence of good mutations*.

Type	*Number*	*Classes*
E_6	67	6
E_7	416	14
E_8	1574	15

Cluster-tilted algebras of Dynkin type A

- Description of the quivers
  [Buan-Vatne 2008, Caldero-Chapoton-Schiffler 2006]
- Complete derived equivalence classification [Buan-Vatne 2008]
- Counting the number of quivers [Torkildsen 2008]

Type	*Number*	*Classes*
A_n	$\sim \frac{1}{\sqrt{\pi}} 4^{n+1} n^{-5/2}$	$\sim \frac{1}{2} n$

Conceptual explanation

A *necessary* condition for

$$\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q), \qquad \big(\text{equivalently, } \Lambda_Q \simeq \mu_k^+(\Lambda_{\mu_k(Q)})\big)$$

is that *both* algebra mutations $\mu_k^-(\Lambda_Q)$ and $\mu_k^+(\Lambda_{\mu_k(Q)})$ are defined.

Theorem [L]. This condition is also *sufficient*!

- That is, if both algebra mutations are defined, they automatically take the correct values.
- Based on a result of [Hu-Xi].

Remark. With slight modifications, applicable to arbitrary cluster-tilted algebras and even more generally, to 2-CY-tilted algebras.

Algorithm to decide on good mutation

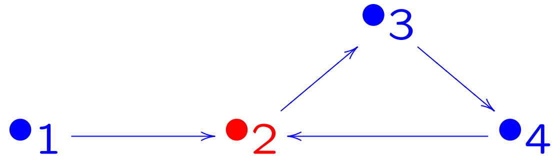
Assume: the Cartan matrices C_{Λ_Q} and $C_{\Lambda_{\mu_k(Q)}}$ are invertible over $\mathbb{Q}$.

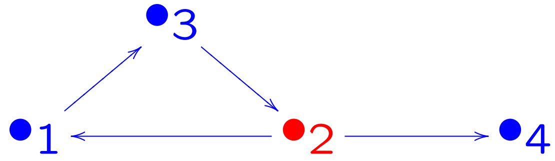
Theorem [L]. There is an effective *algorithm* that decides whether $\Lambda_{\mu_k(Q)} \simeq \mu_k^-(\Lambda_Q)$, using only the data of the Cartan matrices.

It builds on the *Gorenstein* property [Keller-Reiten] and on [Dehy-Keller].

Algorithm – Example

$\Lambda =$





$= \Lambda'$

$$C_\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = C_{\Lambda'}$$

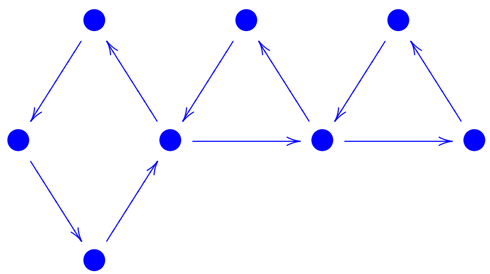
$$C_\Lambda C_\Lambda^{-T} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} = (C_{\Lambda'} C_{\Lambda'}^{-T})^{-1}$$

Sequences of good mutations

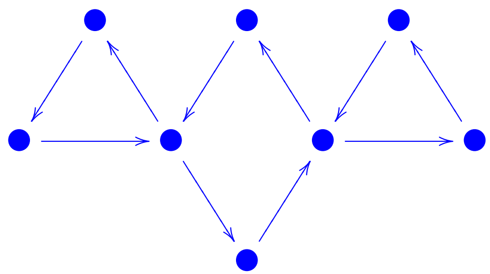
The quivers of derived equivalent cluster-tilted algebras of Dynkin type A or E are connected by sequences of good mutations.

Result [Bastian-Holm-L]. Far-reaching derived equivalence classification of cluster-tilted algebras of Dynkin type D.

Remark. There are derived equivalent cluster-tilted algebras of type D whose quivers are *not* connected by good mutations.



$\sim$



Summary

We discussed the *derived equivalence* of algebras arising from combinatorial data as *quivers with relations*.

- Global reasonings – based on *tensor products*.
- Local reasonings – based on *mutations* of algebras.
- Mutation of algebras vs. *quiver mutation* –
  - Algebras of *global dimension 2*,
  - *2-CY-tilted* algebras, in particular *cluster-tilted* algebras.

For further details, see:
`arXiv:0911.5137`, `arXiv:1001.4765`, `arXiv:0906.3422`.