# Symplectic cohomology of Liouville sectors, part II

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## Outline of parts

I review of past talks
- geometric setup, goals for today,
- motivation for simplex formalism
- conditions on Floer data
- moduli spaces

II algebraic framework for $SC^\bullet(X, \partial X)$
- (filtered) $\infty$-categories
- homotopy coherent diagrams
- homotopy colimits

III construction & properties of $SC^\bullet$, $SH^\bullet$
- construction of $SC^\bullet$
- functoriality
- properties
- canonical map $H^\bullet \longrightarrow SH^\bullet$

# I Review

**Setup** $(X, \partial X, \lambda)$ Liouville sector

$I: \partial X \to \mathbb{R}$ choice of $\frac{1}{2}$-defining fn

$\rightsquigarrow$ product decomp. $\mathrm{Nbd}(\partial X)^{\mathcal{Z}} \simeq \mathrm{Nbd}\, \partial(F \times \mathbb{C}_{\mathrm{Re}z \geq 0})$

$\rightarrow$ projection $\pi: \mathrm{Nbd}(\partial X)^{\mathcal{Z}} \longrightarrow \mathbb{C}_{\mathrm{Re}z \geq 0}$

$\partial X \longrightarrow i\mathbb{R}$

$$\begin{tikzcd} \mathrm{Nbd}\,\partial X \arrow[r, "\simeq"] \arrow[dr, "\pi"'] & \mathrm{Nbd}\,\partial(F \times \mathbb{C}_{\mathrm{Re}\geq 0}) \arrow[d] \\ & \mathbb{C}_{\mathrm{Re}z \geq 0} \end{tikzcd}$$

$\exists$ cylindrical comp. acs $J$ on $X$ s.t. $\pi$ is $J$-holo

goals

- define symplectic cohomology group $SH^\bullet(X, \partial X)$
- show: $SH^\bullet$ is (covariantly) functorial wrt inclusion of L. sectors
- choice of $\pi \rightsquigarrow$ cochain complex $SC^\bullet(X, \partial X)_\pi$ computing $SH^\bullet(X, \partial X)$ functorial in $X$ and $\pi: \mathrm{Nbd}(\partial X)^{\mathcal{Z}} \longrightarrow \mathbb{C}$

motivation for simplex formalism

recall: for Liouville manifolds – 0-simplex

reg. pair $(H, J) \longrightarrow SC^\bullet(H, J)$ ; $SH^\bullet(H, J) := H^\bullet(SC^\bullet(H, J), \partial)$

homotopy $(H_s, J_s)$ from $(H_1, J_1)$ to $(H_2, J_2)$ yields continuation maps ← – – 1-simplex

$\varphi_{12}: SC(H_1, J_1) \longrightarrow SC(H_2, J_2)$

$(\varphi_{12})_*: SH(H_1, J_2) \longrightarrow SH(H_2, J_2)$

$$\begin{tikzcd} & SH(H_2, J_2) \arrow[dr, "(\varphi_{23})_*"] & \\ SH(H_1, J_1) \arrow[ur, "(\varphi_{12})_*"] \arrow[rr, "(\varphi_{13})_*"'] & & SH(H_3, J_3) \end{tikzcd}$$

$\rightsquigarrow$ directed set of $(H, J)$, direct limit $SH(X) = \varinjlim SH^\bullet(H, J)$

L. sectors:

⚠ maps on $SC^\bullet$ are not functorial!

but: two choices of cont. maps are chain homotopic, – 2-simplex

comp. of 2 cont. maps is " to a cont. map – 3-simplex

comp. of such chain htpies is " " chain htpy, etc

$\rightsquigarrow$ homotopies of higher order, simplicial sets

## Conditions on Floer data — details later

form a simplicial set: simplices of any order, in a compatible way

Floer data $(H, J)$ must be

- adapted $\implies$ curves don't pass through bd. $\rightsquigarrow d^2=0$, functoriality
- admissible $\implies$ control orbits near $\infty$ (substitute for "H linear")
- dissipative $\implies$ allow using monotonicity + bd. geometry arguments $\rightarrow$ a priori $C^0$ estimates for cont. maps compactness

## Moduli space setup

domain moduli space $\mathcal{M}^{sc}_n := \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 \geq \ldots \geq a_n\}/\mathbb{R}$

intuition: "curves on a simplex", made rigorous via Morse flow lines

configuration space = universal curve $\mathcal{C}^{sc}_n := \{(p_1, \ldots, p_n) \in (\mathbb{R}\times S^1)^n : \exists s \in S^1 \text{ s.t. } p_i = (a_i, s) \text{ with } a_1 \geq \ldots \geq a_n\}$

can. projection $\mathcal{C}^{sc}_n \longrightarrow \mathcal{M}^{sc}_n$ ; $\mathbb{R}\times S^1$-fibration

compactify:

$$\overline{\mathcal{M}}^{sc}_n = \begin{cases} pt/\mathbb{R} & n=0 \\ [0,\infty]^{n-1} & n \geq 1, \end{cases}$$

$$\overline{\mathcal{C}}^{sc}_n = \begin{cases} S^1 & \text{if } n=0 \\ [0,\infty]^{n-1} \times \mathbb{R}\times S^1 & \text{if } n \geq 1. \end{cases}$$

$n$ = dim. of simplex = max. # levels/pieces allowed

n-simplex of Floer data is a pair of maps

$$H : \overline{\mathcal{C}}^{sc}_n \longrightarrow \mathcal{H}(X) = \{H : X \to \mathbb{R}\}$$

$$J : \overline{\mathcal{C}}^{sc}_n \longrightarrow \mathcal{J}(X) = \{\text{compatible cyl. acs } J \text{ on } X\},$$

induces lower-dimensional simplices

$X_0 \subseteq \ldots \subseteq X_r$ chain of Liouville sectors $\rightsquigarrow$ simplicial set $\mathcal{HJ}_\bullet(X_0, \ldots, X_r)$

n-simplex in $\mathcal{HJ}_\bullet(X_0, \ldots, X_r)$ is an n-simplex in $X_r$ satisfying admissibility, adaptedness, dissipation conditions

**moduli spaces** for $(H,J) \in \mathcal{HJ}_n(X)$, periodic orbits $\gamma^\pm : S^1 \to X$ of $H_0$, $H_1$

$$\mathcal{M}_n(H,J,\gamma^+,\gamma^-) = \left\{ (a_1,\dots,a_n,u) \;\middle|\; \begin{array}{l} a_1 \geq \dots \geq a_n \in \mathbb{R}, \\ u : \mathbb{R}\times S^1 \to X \text{ Floer eq.} \\ u(+\infty,t) = \gamma^+(t),\ u(-\infty,t) = \gamma^-(t) \end{array} \right\} \Big/ \mathbb{R}$$

translate both $(a_i)$ and domain of $u$

(pull back $(H,J)$ from $\overline{\mathcal{C}}^{sc}_n$ to $\mathbb{R}\times S^1$ using choice of $(a_n)$),

compactification $\overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-)$

<u>Prop</u> The moduli spaces $\overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-)$ are compact. $\square$

denote $\mathcal{HJ}^{reg}_\bullet := \{ (H,J) \in \mathcal{HJ}_\bullet \mid \text{all } \overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-) \text{ are cut out transversely} \}$,

for any map $(\Delta^k, \partial\Delta^k) \to (\mathcal{HJ}_\bullet, \mathcal{HJ}^{reg}_\bullet)$,
can perturb $J$-component to a map $\Delta^k \to \mathcal{HJ}^{reg}_\bullet$.

$\to$ $\mathcal{HJ}^{reg}_\bullet$ is a **filtered $\infty$-category**,
each inclusion $\mathcal{HJ}^{reg}_\bullet \hookrightarrow \mathcal{HJ}_\bullet$ is **cofinal**.

# II Algebraic framework

## Filtered ∞-categories

simplex ⇢ simplex category ⇢ simplicial set ⇢ nerve of a category

simplicial set ⇣ ∞-category ⇠ nerve of a category

(filtered ∞-category) ⇠ ∞-category

geometrically: **n-simplex** is $\Delta^n = \{ \underline{x} \in [0,1]^n : 0 \le x_1 \le \dots \le x_n \le 1 \}$ GPS

$\cong \{ \underline{x} \in [0,1]^{n+1} : \sum x_i = 1 \}$ Lurie


essence: n-simplex has n+1 vertices, inductive definition

algebraically: **n-simplex** is the totally ordered set $[n] = (\{0, \dots, n\}, \le)$

**simplex category** $\Delta$

objects are the sets $[n]$

morphisms are order-preserving fns ⇠ $f : [n] \to [m]$, $a \le b \Rightarrow f(a) \le f(b)$

a **simplicial set** is a contravariant functor $\Delta \xrightarrow{K} \mathrm{Sets}$,

i.e. collection $\{K_n\}_{n \in \mathbb{N}}$ of sets

each $[n] \xrightarrow{f} [m]$ induces a fn $K(f) : K_m \to K_n$, comp. with composition.

"simplicial map"
Map $K \to L$ of simplicial sets is a natural transformation $K \to L$,

i.e. collection of maps $\{K_n \xrightarrow{\phi_n} L_n\}_{n \in \mathbb{N}}$

$$\begin{tikzcd} K_m \arrow[r, "K(f)"] \arrow[d, "\phi_m"] & K_n \arrow[d, "\phi_n"] \\ L_m \arrow[r, "L(f)"] & L_n \end{tikzcd}$$

s.t. for any $f : [n] \to [m]$ order-preserving,

the induced maps $K(f) : K_m \to K_n$, $L(f) : L_m \to L_n$ commute with the $\{\phi_n\}$.

The nerve of a category $\mathcal{C}$ is the simplicial set $N\mathcal{C}$
with sets $N\mathcal{C}_n = \{\text{functors } [n] \to \mathcal{C}\}$

$$= \{\text{seq. of } n \text{ composable morphisms } C_0 \xrightarrow{f_1} C_1 \to \dots \xrightarrow{f_n} C_n\},$$

maps are compositions of face and degeneration maps.

face map $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ induced by un. map $[n-1] \to [n]$
"skipping $i$", i.e. $i-1 \mapsto i-1$, $i \mapsto i+1$ etc.

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} C_n \overset{d_i}{\longmapsto} C_0 \to \dots \to C_{i-1} \xrightarrow{f_{i+1}\circ f_i} C_{i+1} \xrightarrow{f_{i+2}} \dots \to C_n$$

degeneracy map $s_j : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ induced by un. map $[n+1] \to [n]$
"doubling $j$", i.e. $j \mapsto j$, $j+1 \mapsto j$ etc.

$$C_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} C_n \overset{s_j}{\longmapsto} C_0 \xrightarrow{f_1} \dots \xrightarrow{f_i} C_i \xrightarrow{id} C_i \xrightarrow{f_{i+1}} C_{i+1} \to \dots \to C_n.$$

Prop injective correspondence

$$\text{(small) categories} \longrightarrow \text{simplicial sets}$$
$$\mathcal{C} \longmapsto N\mathcal{C}. \qquad \square$$

[Inverse map is straightforward, e.g. $Ob(\mathcal{C}) \triangleq N\mathcal{C}_0$.]

(Fact $\exists$ nice char. of s. sets which are nerves of a category.)

An $\infty$-category is a simplicial set $K$
with the extension property over $\Lambda^n_i \hookrightarrow \Delta^n$ for $0<i<n$,
where $\Lambda^n_i = \Delta^n \setminus (\text{face opp. vertex } i \cup \text{int } \Delta^n)$
is the $i$-th horn.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Small print: $\Delta^n$ is a simplicial set,
$\Delta^n_i := \mathrm{Hom}_\Delta([i],[n]) = \{\text{ord.-preserving } [i] \to [n]\}$; obvious morphism
each face of $\Lambda^n_i$ is a simplicial set,
so $\Lambda^n_i \longrightarrow K$ means a collection of simplicial maps compatible with each other
diagram of simplicial maps.

Fact. Every nerve $N\mathcal{C}$ is an $\infty$-category.

intuition: can compose (higher) morphisms up to homotopy

example, $n=2$


$\Lambda^2_1 \mathrel{\hat{=}}$ composition of morphisms $0\to 1$, $1 \to 2$

edge $0-2 \mathrel{\hat{=}}$ morphism $0 \to 2$

extension $\mathrel{\hat{=}}$ homotopy $0 \to 1 \to 2 \sim 0 \to 2$

general

faces containing $i = \Lambda^n_i \mathrel{\hat{=}}$ (higher) morphisms to compose

opposite face $\mathrel{\hat{=}}$ 〃 equiv. to composition

filled simplex $\mathrel{\hat{=}}$ homotopy between these

Skip definition of a filtered $\infty$-category

$K$ simplicial set $\rightsquigarrow K^{\triangleleft} = K$ with an initial vertex added

$K^{\triangleright} =$ 〃 a terminal 〃

$\rightsquigarrow$ under-category $\mathcal{C}_{c/}$,
over-category $\mathcal{C}_{/c}$
for an $\infty$-category $\mathcal{C}$ and $c \in \mathcal{C}_0$

$\rightsquigarrow$ filtered $\infty$-category, cofinal functor between (filtered) $\infty$-categories

$\exists$ simple definition of filtered (ordinary) categories, cf. GPS, §3.4

# Homotopy coherent diagrams

denote $Ch := Ch^{\mathbb{Z}_2}$ = category of $\mathbb{Z}_2$-graded cochain complexes of abelian groups

a diagram of chain complexes is a map $K \to Ch$ of simplicial sets
identify with its nerve

i.e. maps $K_n \to N\,Ch_n = \{ A_0^\bullet \to A_1^\bullet \to \dots \to A_n^\bullet$ seq. of chain maps$\}$

compatible with induced maps $K_n \to K_m$, $N\,Ch_n \to N\,Ch_m$.

denote by $C_\bullet(\mathcal{F}(\Delta^n))$ the cubical chain complex of $\mathcal{F}(\Delta^n) = [0,1]^{n-1}$,

with chain groups $C_k(\mathcal{F}(\Delta^n)) = \mathbb{Z}\langle k\text{-faces of } \mathcal{F}(\Delta^n)\rangle$

and standard differential

$\longrightarrow$ is free of rank $3^{n-1}$

recall $\overline{\mathcal{M}}_n^{SC} \simeq \mathcal{F}(\Delta^n)$ = suitable Morse flow lines on $\Delta^n$

the differential graded nerve of $Ch$ is the simplicial set $N_{dg}Ch$,

whose $p$-simplices are $(p+1)$-tuples of chain complexes $A_0^\bullet, \dots, A_p^\bullet \in Ch$

tensor product of cochain complexes

along with chain maps
$$f_\sigma : A^\bullet_{\sigma(0)} \otimes C_{-\bullet}(\mathcal{F}(\Delta^q)) \longrightarrow A^\bullet_{\sigma(q)}$$

for every map $\sigma : \Delta^q \to \Delta^p$ s.t. $(q \geq 1)$

- for $0<k<q$, $\quad f_\sigma|_{k\dots q} \circ f_\sigma|_{0\dots k} = f_\sigma|_{C_{-\bullet}(\mathcal{F}(\Delta^k)) \times C_{-\bullet}(\mathcal{F}(\Delta^{q-k}))}$

  wrt the natural map $\mathcal{F}(\Delta^k) \times \mathcal{F}(\Delta^{q-k}) \longrightarrow \mathcal{F}(\Delta^q)$

  compat. with gluing

- for every $\tau : \Delta^r \to \Delta^q$ with $\tau(0)=0$, $\tau(r)=q$, $(q \geq 1)$

  $f_{\sigma\circ\tau} = f_\sigma \circ \tau_*$ where $\tau_* : \mathcal{F}(\Delta^r) \to \mathcal{F}(\Delta^q)$ induced from $\tau$

  maps between sets.

a homotopy coherent diagram of chain complexes is a map $K \to N_{dg}Ch$.

($\exists$ tautological map $Ch \to N_{dg}Ch$.)

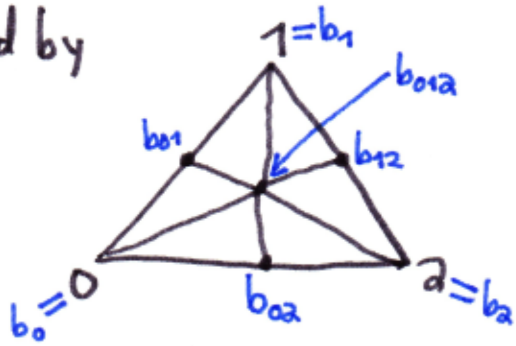
# Homotopy colimit

big picture

$$\text{directed system} \xrightarrow{\text{direct limit}} \text{graded ab. group}$$

$$\text{htpy coh. diagram} \xrightarrow[\text{htpy colimit}]{} \text{(co)chain complex}$$

The **Barycentric subdivision** of a simplex is obtained by forming the **barycentre** $b_\sigma$ of each face $\sigma$ and considering all simplices formed by barycentres of a chain of faces,




i.e. simplices are ordered chains $\{b_{\sigma_1}, \dots, b_{\sigma_q}\}$ where $\sigma_1 \subset \dots \subset \sigma_q$ faces of the simplex.

$\exists$ Barycentric subdivision of a s. set — "opposite" of std. def.

$X_\cdot$ simplicial set $\rightsquigarrow$ **Barycentric (co)subdivision** is the s. set $bX_\cdot$,

a p-simplex of $bX_\cdot$ is a chain $\Delta^{a_p} \hookrightarrow \dots \hookrightarrow \Delta^{a_0} \longrightarrow X_\cdot$ of simplicial sets

natural maps $\quad a_1 \geq \dots \geq a_p$

$\exists$ natural map $r : bX_\cdot \longrightarrow X_\cdot$,

$$\underset{\text{of } bX_\cdot}{\text{p-simplex}} \quad \Delta^{a_p} \hookrightarrow \dots \hookrightarrow \Delta^{a_0} \xrightarrow{\sigma} X_\cdot \longmapsto \sigma(0 \in \Delta^{a_0}, \dots, 0 \in \Delta^{a_p}) \in X_\cdot$$

given a diagram $A : bX_\cdot \longrightarrow N_{dg}Ch$, denote

$$C_\cdot(X_\cdot, A) := \bigoplus_{\sigma : \Delta^n \to X} A(\sigma)[n]$$

"s. chains on $X_\cdot$ with coeff. in $A$"

check: $\sigma : \Delta^n \to X$ is a 0-simplex of $bX_\cdot$

$A(\sigma)$ is a 0-simplex of $N_{dg}Ch$ = chain cpx,

$A(\sigma)[n]$ shifts the complex $A(\sigma)$ by $n$.

for any diagram $A : X_\cdot \longrightarrow N_{dg}Ch$, the **homotopy colimit** of $A$ is

$$\operatorname*{hocolim}_{X_\cdot} A := C_\cdot(X_\cdot; A \circ r).$$

# III Construction and properties of $SC^\bullet$ and $SH^\bullet$

## Construction of $SH^\bullet$ and $SC^\bullet$

step 1 produce a map $\mathcal{H}\mathcal{J}^{reg}_\bullet(X) \longrightarrow N_{dg}Ch_1$

i.e. a diagram of symplectic cochain complexes over $\mathcal{H}\mathcal{J}^{reg}_\bullet(X)$

strategy: count curves in $\overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-)$

vertex $(H,J) \in \mathcal{H}\mathcal{J}^{reg}_0(X) \rightsquigarrow$ Floer complex $CF^\bullet(X;H)$

$$CF^\bullet(X;H) := \bigoplus_{\phi_H(x)=x} \sigma_{\phi_{H},x}$$ ← - - - orientation line (details skipped)

← - - - fixed points of time-1 flow $\phi_H : X \to X$

differential: count spaces $\overline{\mathcal{M}}_0(H,J,\gamma^+,\gamma^-)$ ← - - - 0-dim. ones

compactness, gluing $\longrightarrow CF^\bullet(X;H)$ cochain complex (standard)

1-simplex $(H,J) \in \mathcal{H}\mathcal{J}^{reg}_1(X)$ defines chain map

$$F_{(H,J)} : CF^\bullet(X; H(0)) \longrightarrow CF^\bullet(X; H(1))$$

by counting spaces $\mathcal{M}_1(H,J,\gamma^+,\gamma^-)$ ← - - - - - 0-dim. ones

n-simplex $(H,J) \in \mathcal{H}\mathcal{J}^{reg}_n(X)$, $n \geq 1$ $\rightsquigarrow \overline{\mathcal{M}}^{sc}_n = \mathcal{F}(\Delta^n)$ is a cube

$\rightsquigarrow$ count 0-dim. components of inverse image

of any of the $3^{n-1}$ strata of $\overline{\mathcal{M}}^{sc}_n$ in $\overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-)$

wrt $\overline{\mathcal{M}}_n(H,J,\gamma^+,\gamma^-) \to \overline{\mathcal{M}}^{sc}_n$, $[(u,a_1,\dots,a_n)] \longmapsto [(a_1,\dots,a_n)]$

$\rightsquigarrow$ chain map

$$F_{(H,J)} : CF^\bullet(X; H(0)) \otimes C_{-\bullet}(\mathcal{F}(\Delta^n)) \longrightarrow CF^\bullet(X; H(n))$$

has degree zero by inspection

e.g. $n=1$: continuation maps $CF^\bullet(X; H(0)) \longrightarrow CF^\bullet(X; H(1))$ preserve grading

Lemma. This defines a diagram $\mathcal{H}\mathcal{J}^{reg}_\bullet(X) \longrightarrow N_{dg}Ch$.

Proof. Only one non-trivial condition to check:

for any degenerate $(n+1)$-simplex $(H',J')$,

$$F_{(H',J')}(- \otimes [\mathcal{F}(\Delta^{n+1})]) = \begin{cases} 0 & n>0 \\ \mathrm{id} & n=0, \end{cases}$$

where $[\mathcal{F}(\Delta^{n+1})]$ is the top-dim. generator of $C_{-\bullet}(\mathcal{F}(\Delta^{n+1}))$.

say, $(H', \partial') =$ pullback of $n$-simplex $(H, \partial)$
under $k_j : \Delta^{n+1} \to \Delta^n$, $\begin{matrix} j \\ j+1 \end{matrix} \longmapsto j \in \Delta^n$.

$\Rightarrow (H', \partial') = \pi_j^* (H, \partial)$, where $\pi_j : \overline{\mathcal{C}}^{SC}_{n+1} \to \overline{\mathcal{C}}^{SC}_n$ forgets $a_{j+1}$

$\Rightarrow$ almost get a map $\overline{\mathcal{M}}_{n+1}(\pi_j^*(H, \partial), \gamma^+, \gamma^-) \longrightarrow \overline{\mathcal{M}}_n(H, \partial, \gamma^+, \gamma^-)$

$$(a_1, \dots, a_{n+1}, u) \longmapsto (a_1, \dots, \widehat{a_{j+1}}, \dots, a_{n+1}, u)$$

except $(a_1, \dots, \widehat{a_{j+1}}, \dots, a_{n+1}, u)$ might be unstable.

but: $\overline{\mathcal{M}}_{n+1}(\pi_j^*(H, \partial), \gamma^+, \gamma^-)$ contains no split trajectories
$\Rightarrow$ trajectory remains stable for $n \neq 0$
for $n = 0$, an unstable trajectory has an $\mathbb{R}$-symmetry $\rightsquigarrow$ trivial cylinder
$\rightsquigarrow$ contributes id

note $\dim \overline{\mathcal{M}}_{n+1}(\pi_j^*(H, \partial), \gamma^+, \gamma^-) = 0 \Rightarrow \dim \overline{\mathcal{M}}_n(H, \partial, \gamma^+, \gamma^-) = -1$
$\Rightarrow \overline{\mathcal{M}}_n(H, \partial, \gamma^+, \gamma^-) = \emptyset$ $\square$

transversality for generic elements of $\overline{\mathcal{J}}^{SC}_{n+1}$

symplectic cochain complex of $X$ is

$$SC^*(X, \partial X) := \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_*(X)} CF^*(X; -)$$

symplectic cohomology of $X$ is $SH^*(X, \partial X) := H^*(SC^*(X, \partial X))$

<u>Obs</u> $SH^*(X, \partial X) = \varinjlim HF^*(X; H)$, direct limit over any cofinal collection.

<u>"Proof"</u> $\mathcal{H}\mathcal{J}^{reg}_*$ is filtered + abstract nonsense. $\square$

## Functoriality

The diagram over $\mathcal{H}\mathcal{J}^{reg}_\bullet(X)$ generalises to $\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r)$

By any chain $X_0 \subset \dots \subset X_r$ of Liouville sectors.

forgetful maps $\mathcal{H}\mathcal{J}_\bullet(X_0,\dots,X_r) \longrightarrow \mathcal{H}\mathcal{J}_\bullet(X_0,\dots,\hat{X}_i,\dots,X_r)$ of s. sets

preserve $\mathcal{H}\mathcal{J}^{reg}_\bullet$ $\rightsquigarrow$ maps $\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r) \to \mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,\hat{X}_i,\dots,X_r)$

$\xrightarrow{\text{pullback}}$ diagrams $CF^\bullet(X_i;-)$ over $\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r)$

<u>Geometric lemma</u> For $(H,J) \in \mathcal{H}\mathcal{J}_\bullet(X_0,\dots,X_r)$, any trajectory in $\mathcal{M}_n(H,J,\gamma^+,\gamma^-)$ with pos. end $\gamma^+$ in $X_i$ lies entirely inside $X_i$. $\square$

$\rightsquigarrow$ inclusions $CF^\bullet(X_0;-) \subseteq \dots \subseteq CF^\bullet(X_r;-)$

for an inclusion $X \subseteq X'$ of Liouville sectors, consider

$$\operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X)} CF^\bullet(X;-) \xleftarrow{\simeq} \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X,X')} CF^\bullet(X;-) \longrightarrow \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X')} CF^\bullet(X';-)$$

forgetful map $\mathcal{H}\mathcal{J}^{reg}_\bullet(X,X') \to \mathcal{H}\mathcal{J}^{reg}_\bullet(X)$ is cofinal $\Rightarrow$ left map quasi-iso

$\rightsquigarrow$ map $SH^\bullet(X,\partial X) \longrightarrow SH^\bullet(X',\partial X')$

functoriality of $SC^\bullet$

consider $$\overbrace{\operatorname*{hocolim}_{\substack{X_0 \subseteq \dots \subseteq X_r \\ X_i \text{ L.sectors}}} \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r)} CF^\bullet(X_0,-)}^{(*)},$$

is clearly functorial wrt inclusion of Liouville sectors

show $SC^\bullet(X,\partial X) \simeq (*)$

forgetful maps $\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r) \longrightarrow \mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,\hat{X}_i,\dots,X_r)$

induce natural maps

$$\operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r)} CF^\bullet(X_0;-) \longrightarrow \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,\hat{X}_i,\dots,X_r)} CF^\bullet(X_0;-) \quad \text{for } i>0 \qquad (**)$$

$$\operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X_0,\dots,X_r)} CF^\bullet(X_0;-) \longrightarrow \operatorname*{hocolim}_{\mathcal{H}\mathcal{J}^{reg}_\bullet(X_1,\dots,X_r)} CF^\bullet(X_1;-)$$

forgetful map cofinal $\xrightarrow[\text{nonsense}]{\text{abstract}}$ $(**)$ quasi-iso

$\longrightarrow$ inclusion $SC^\bullet(X,\partial X) \hookrightarrow (*)$ $r=0, X_0=X$ is a quasi-iso.

## Properties

**Prop** If $X \subseteq X'$ is a trivial inclusion of Liouville sectors,
the induced map $SH^\bullet(X,\partial X) \longrightarrow SH^\bullet(X',\partial X')$ is an isomorphism. $\square$

**Cor** $SH^\bullet(X,\partial X)$ is invariant (up to canonical iso) under deformation of $X$.

**Proof** An arbitrary deformation is a composition
of triv. inclusions and their inverses. $\square$

**Conjecture** $SH^\bullet$ satisfies a Künneth formula
$\exists$ natural quasi-iso $SC^\bullet(X,\partial X) \times SC^\bullet(X',\partial X') \longrightarrow SC(X \times X', \partial(X \times X'))$.

## Canonical map $H^\bullet \longrightarrow SH^\bullet$

**Recall** When $H, J$ are $S^1$-invariant,
Floer trajectory with $\partial_t u \equiv 0$ = Morse trajectory for $H$ wrt $\omega(\cdot, J\cdot)$
$\rightarrow$ map $\{$Morse trajectories$\} \longrightarrow \{$Floer trajectories$\}$.

**Prop** Let $(H,J) \in \mathcal{HJ}_n(X)$ be $S^1$-invariant with $H(i)$ Morse, $i \in \Delta^n$ vertices
s.t. all Morse trajectories are cut out transversely.
For $\delta > 0$ suff. small, the map
$\{$Morse traj. of $H$ wrt $\omega(\cdot, J\cdot)\} \longrightarrow \{$Floer traj. for $(\delta \cdot H, J)\}$
is bijective and $(\delta \cdot H, J) \in \mathcal{HJ}_n^{reg}(X)$. $\square$

**Prop** $\exists$ canonical map $H^\bullet(X,\partial X) \longrightarrow SH^\bullet(X,\partial X)$,
is functorial wrt inclusions of Liouville sectors. $\square$

**Prop** If $\partial_\infty X$ admits a cutoff Reeb vector field with no periodic orbits,
the natural map $H^\bullet(X,\partial X) \longrightarrow SH^\bullet(X,\partial X)$ is an iso. $\square$