Exercise Session 8

① Let $R = \varinjlim_{i} R_i$ filtered colimit of rings.

(a) Let $X \to \operatorname{Spec} R$ fin. pres. Then $\exists X_i \to \operatorname{Spec} R_i$ s.t.

$$X = X_i \otimes_{R_i} R$$

• Assume $X = \operatorname{Spec} A$ affine. Then

$$A = R[T_1, \dots, T_n]/(f_1, \dots, f_m)$$

Then for some i, all coefficients of all f_k's lie in R_i.

$\leadsto A_i := R_i[T_1, \ldots, T_n]/(f_1, \ldots, f_m)$

$$\Rightarrow A \cong A_i \underset{R_i}{\otimes} R \qquad \checkmark$$

- Let X be general. Let $X = U^1 \cup \ldots U^n$ with all U^k affine.

$$\leadsto U^k = U_i^k \underset{R_i}{\otimes} R$$

for some $i \in I$, U_i^k over $\operatorname{Spec} R_i$.

Let $U^{kk'} := U^k \cap U^{k'}$. Then $U^{kk'}$ is quasicompact, hence

$$U^{kk'} = U^k \setminus V(a_1, \ldots, a_d)$$

for some $a_1, \ldots, a_d \in \mathcal{O}_{U^k}(U^k)$.

$\leadsto a_1, \ldots, a_d$ already defined over some R_i

$\Rightarrow$ After enlarging i, can find

$$\mathcal{U}_i^{kk'} \subseteq \mathcal{U}_i^k \quad \text{s.t. } \mathcal{U}_i^{kk'} \otimes_{R_i} R = \mathcal{U}^{kk'}$$

By (G), the isom.

$$\begin{tikzcd} \mathcal{U}^{kk'} \arrow[r, bend left, "id"] & \mathcal{U}^{k'k} \arrow[l, bend left, "id"] \end{tikzcd}$$

comes from an isom.

$$\begin{tikzcd} \mathcal{U}_i^{kk'} \arrow[r, bend left, "\varphi_i^{kk'}"] & \mathcal{U}_i^{k'k} \arrow[l, bend left, "\varphi_i^{k'k}"] \end{tikzcd}$$

(after enlarging i).

Check: $\left((\mathcal{U}_i^k)_k, (\mathcal{U}_i^{kk'})_{k,k'}, (\varphi_i^{kk'})_{k,k'} \right)$

is a glueing datum (after increasing i):

- $(\varphi_i^{kk'})^{-1}(U_i^{k'k} \cap U_i^{k'k''})$

$$= U_i^{kk'} \cap U_i^{kk''}$$

- $$\begin{tikzcd} U_i^{kk'} \cap U_i^{kk''} \arrow[rr, "\varphi_i^{kk''}"] \arrow[dr, "\varphi_i^{kk'}"'] & \circlearrowleft & U_i^{kk'} \cap U_i^{k''k'} \\ & U_i^{k'k} \cap U_i^{k'k''} \arrow[ur, "\varphi_i^{k'k''}"'] & \end{tikzcd}$$

(Use (G)).

$\Rightarrow$ Glue this to get a scheme X_i over $\operatorname{Spec} R_i$. Clearly $X_i \otimes_{R_i} R = X$.

(b) $\varinjlim_i \operatorname{Hom}_{R_i}(X_i, Y_i) \xrightarrow{\sim} \operatorname{Hom}_R(X, Y)$

(where $X_i = X_0 \otimes_{R_0} R_i$, same for Y_i)

$$X = X_0 \otimes_{R_0} R$$

<u>Injectivity</u>: Let $f_i, g_i : X_i \to Y_i$ s.t. $f, g : X \to Y$ are equal.

Let $Y_i = V_i^1 \cup \dots \cup V_i^n$ with all V_i^k affine. Then $Y = V^1 \cup \dots \cup V^n$ and $f^{-1}(V^k) = g^{-1}(V^k)$.

$\longrightarrow$ after enlarging i,

$$g_i^{-1}(V_i^k) = f_i^{-1}(V_i^k)$$

(wlog X_i affine

$\rightsquigarrow f_i^{-1}(V_i^k) = X_i \setminus V(a_1, \dots, a_m)$

$\rightsquigarrow g_i^{-1}(--) - - \_\_$

see Stacks Lemma 01Z4(2))

$\Rightarrow$ wlog Y_i affine.

$\rightsquigarrow$ easily reduce to X_i affine as well

Rest is easy to check (commutative algebra)

<u>Surjectivity</u>: same ideas as above.

<u>Rmk</u>: (a) + (b):

$$\mathrm{FinPresSch}/_R \cong \varinjlim_i \mathrm{FinPresSch}/_{R_i}$$

(c) k field, $E, E'/k$ EC. For some fin. k'/k we have

$$\mathrm{Hom}(E_{\bar{k}}, E'_{\bar{k}}) = \mathrm{Hom}(E_{k'}, E'_{k'}).$$

By (6)

$$\mathrm{Hom}(E_{\bar{k}}, E'_{\bar{k}}) = \varinjlim_{\substack{k'/k \\ \text{finite}}} \mathrm{Hom}(E_{k'}, E'_{k'})$$

But LHS is fin. gen. $/\mathbb{Z}$.

$$\rightarrow \mathrm{Hom}(E_{k'}, E'_{k'}) \twoheadrightarrow \mathrm{Hom}(E_{\bar{k}}, E'_{\bar{k}})$$

for some k'.

Injectivity done in lecture.

(c) $k=\bar{k}$, L any field ext of k.

(a) Show that

$$\mathrm{Hom}_k(\mathbb{P}^1_k, \mathbb{P}^1_k) \subsetneq \mathrm{Hom}_L(\mathbb{P}^1_L, \mathbb{P}^1_L)$$

$$\begin{tikzcd}
\mathrm{Hom}_k(k(t),k(t)) \arrow[rr, hook] \arrow[d, phantom, "\shortparallel"] & & \mathrm{Hom}_L(L(t),L(t)) \arrow[d, phantom, "\shortparallel"] \\
k(t) & & L(t) \\
\{\text{non-constant maps } \mathbb{P}^1_k \to \mathbb{P}^1_k\} \arrow[uu, bend left, "\shortparallel"] & & \{\text{non-constant maps } \mathbb{P}^1_L \to \mathbb{P}^1_L\} \arrow[uu, bend right, "\shortparallel"'] \\
\cup\!\mid & & \cup\!\mid \\
\{\text{non-constant maps } \mathbb{P}^1_k \to \mathbb{P}^1_k \text{ fixing } 0\} & & \{\text{non-constant maps } \mathbb{P}^1_L \to \mathbb{P}^1_L \text{ fixing } 0\} \\
\cup\!\mid & & \cup\!\mid \\
t \mapsto at,\ a \in k^\times & \subsetneq & t \mapsto at,\ a \in L^\times
\end{tikzcd}$$

(6) Let $E, E' \in C/k$. Then

$$\mathrm{Hom}(E, E') \xrightarrow{\sim} \mathrm{Hom}(E_L, E'_L).$$

By 1.(6),

$$\mathrm{Hom}(E_L, E'_L) = \varinjlim_{\substack{R/k \\ \text{fin type.}}} \mathrm{Hom}(E_R, E'_R)$$

$\rightsquigarrow$ Can replace L by R, i.e. want

$$\mathrm{Hom}(E, E') \xrightarrow{\sim} \mathrm{Hom}(E_R, E'_R)$$

Let $x \in \mathrm{Spec}\, R$ be a closed point.

section of $\mathrm{Spec}\, R \to \mathrm{Spec}\, k$

• <u>Injectivity</u>: Suppose $\varphi : E \to E'$ is s.t. $\varphi_R = 0$.

$$\Rightarrow \varphi = (\varphi_R)_x = 0.$$

Surjectivity: Let $\varphi : E_R \to E_R$. Let

$$\varphi_0 := (\varphi_x)_R .$$

Consider

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, hook] \arrow[dr, "\mathrm{id}"'] & \operatorname{Spec} R \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

$$f := \varphi - \varphi_0 : \underset{X}{E} \times \underset{Y}{\operatorname{Spec} R} \to \underset{Z}{E'}$$

Apply rigidity (lecture 6)

$$\Rightarrow \varphi - \varphi_0 = 0.$$

3) $L = \varinjlim_{n \in \mathbb{Z}} \mathcal{O}_L [n^{-1}]$.

(a) Use Weierstraß equ.

(b) Use 1. (b)

(c) Sheet 7.